

# Mass spectroscopy using Borici-Creutz fermion on 2D lattice

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(Dated: March 15, 2016)

## Abstract

Minimally doubled fermion proposed by Creutz and Borici is a promising chiral fermion formulation on lattice. In this work, we present excited state mass spectroscopy for the meson bound states in Gross-Neveu model using Borici-Creutz fermion. We also evaluate the effective fermion mass as a function of coupling constant which shows a chiral phase transition at strong coupling. The lowest lying meson in 2-dimensional QED is also obtained using Borici-Creutz fermion.

PACS numbers: 11.15.Ha, 11.10.Kk, 11.30.Rd, 12.40Yx

## I. INTRODUCTION

Chiral fermion formulation is always a challenging task on lattice and minimally doubled fermions in recent times have drawn attention as promising lattice formulations of chiral fermion. Karsten[1] and Wilczek[2] proposed one formulation of minimally doubled fermion and another was developed by Creutz[3] and Borici[4]. Both the formulations break the hypercubic symmetry on the lattice [5] and thus allow non-covariant counter terms. The important question is how bad the effects of the symmetry breaking are in a numerical simulation. It was shown that a consistent renormalizable theory for minimally doubled fermion can be constructed by fixing only three counter terms allowed by the symmetry and the counter terms for BC action at one loop in perturbation theory have been evaluated [6, 7]. But, till date, sufficient numerical studies of the minimally doubled fermions have not been done. The purpose of this work is to investigate Borici-Creutz(BC) formulation numerically in some models. The BC fermion formulation was motivated by the fact that electrons on graphene lattice are described by a massless quasi-relativistic Dirac equation. The BC fermion describes two chiral modes or the two flavors of chiral fermion located at  $(0, 0, 0, 0)$  and at  $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ . It was shown that in presence of gauge background with integer-valued topological charge, BC action satisfies the Atiyah-Singer index theorem[8]. In [9], using BC fermion we have shown a chiral phase transition in the Gross-Neveu model. We extend that investigation further in this work to meson spectroscopy of the Gross-Neveu model as well as in 2D QED (QED<sub>2</sub>) using BC fermion. Chiral and parity-broken(Aoki) phase structures of the Gross-Neveu model have been studied for Wilson and Karsten-Wilczek fermions [10, 11]. A lattice simulation of the Gross-Neveu model using the Wilson fermion was done by Korzec et al[12], where the recovery of chirally invariant Gross-Neveu model from a lattice model was studied. The semimetal-insulator phase transition on a graphene lattice with Thirring type four fermion interactions has been studied by Hands and collaborators[13] and the strong coupling analysis of the tight-binding graphene model with Kekule distortion term has been done by Araki[14].

In this paper, we perform hybrid Monte Carlo(HMC) simulation to investigate the excited state spectrum of the lattice Gross-Neveu model. Extraction of the excited state spectrum is always a difficult task on lattice. Till date, variational method gives the best spectrum. From the slope of the correlators, we first obtain some preliminary estimate about the masses

and then we use variational method to extract the meson masses. Excited state spectroscopy in the Gross-Neveu model with Wilson fermion has been studied in [15], where the authors obtained the ground state as the only bound state and the other excited states were scattering states. With the BC fermion, we have obtained three states, two of them are bound states (ground state and one excited state) and the third one appears to be a scattering state. We also evaluate the fermion mass in the model which shows that it is consistent with a chiral phase transition at large coupling observed in [9]. Next we investigate the meson mass spectrum in QED in two dimension. QED<sub>2</sub> having confinement serves as a toy model for QCD and hence QED<sub>2</sub> or Schwinger model has been studied to great extent on lattice (see [16, 17] and references therein). Schwinger model using Hamiltonian formalism on lattice has been investigated in [18]. QED<sub>2</sub> also serves as a good toy model for numerical study of chiral fermions. A 2-flavor Schwinger model with light fermions have been studied with dynamical overlap fermion [19, 20]. Here, we study the model with the minimally doubled fermion, namely, the BC fermion.

## II. SPECTROSCOPY OF THE GROSS-NEVEU MODEL

The free BC action in 2D is written as,

$$S = \sum_n \left[ \frac{1}{2} \sum_\mu \bar{\psi}_n \gamma_\mu (\psi_{n+\mu} - \psi_{n-\mu}) - \frac{ir}{2} \sum_\mu \bar{\psi}_n (\Gamma - \gamma_\mu) (2\psi_n - \psi_{n+\mu} - \psi_{n-\mu}) - i(2 - c_3) \bar{\psi}_n \Gamma \psi_n + m_0 \bar{\psi}_n \psi_n \right], \quad (1)$$

where,  $\mu = 1, 2$  and  $\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2)$  satisfies  $\{\Gamma, \gamma_\mu\} = 1$ . Including four-fermion interactions, the Gross-Neveu model on lattice is given by

$$S = \sum_n \left[ \frac{1}{2} \sum_\mu \bar{\psi}_n \gamma_\mu (\psi_{n+\mu} - \psi_{n-\mu}) - \frac{ir}{2} \sum_\mu \bar{\psi}_n (\Gamma - \gamma_\mu) (2\psi_n - \psi_{n+\mu} - \psi_{n-\mu}) - i(2 - c_3) \bar{\psi}_n \Gamma \psi_n + m_0 \bar{\psi}_n \psi_n - \frac{g^2}{2N} [(\bar{\psi}_n \psi_n)^2 + (\bar{\psi}_n i\Gamma \psi_n)^2] \right], \quad (2)$$

where,  $g$  is the coupling constant which we consider the same for both four point (scalar and vector) interactions and we set  $r = 1$  in our calculations. Since the parity is broken by the BC action, a counter term  $c_3$  is added to it. Detailed discussion about the  $c_3$ -term can be found in [9]. Now, the action is rewritten explicitly in terms of the auxiliary fields as

$$S = \sum_{m,n} \bar{\psi}_m M_{mn} \psi_n + \frac{N}{2g^2} (\sigma^2 + \pi_\Gamma^2), \quad (3)$$

where  $N$  is the number of flavors. The auxiliary fields

$$\begin{aligned}\sigma &= -\frac{g^2}{N}(\bar{\psi}\psi), \\ \pi_\Gamma &= -\frac{g^2}{N}(\bar{\psi}i\Gamma\psi)\end{aligned}\tag{4}$$

are defined in the dual lattice sites  $\tilde{x}$  surrounding the direct lattice site  $x$  [21].

$$M_{mn} = D_{mn} + \frac{1}{4} \sum_{\langle x, \tilde{x} \rangle} (\sigma(\tilde{x}) + i\pi_\Gamma(\tilde{x})\Gamma),\tag{5}$$

where  $\langle x, \tilde{x} \rangle$  denotes four dual lattice sites  $\tilde{x}$  surrounding the direct lattice site  $x$  and  $D_{mn}$  is the BC Dirac operator:

$$D_{mn} = \frac{1}{2}\gamma_\mu(\delta_{n,m+\mu} - \delta_{n,m-\mu}) + \frac{i}{2}(\Gamma - \gamma_\mu)(\delta_{n,m+\mu} + \delta_{n,m-\mu}) - ((2 - c_3)i\Gamma - m_0)\delta_{m,n}.\tag{6}$$

Since  $M$  is a complex matrix, we work with  $(M^\dagger M)$  to make it real and positive definite and integrate out the fermion fields by the pseudofermion method. Since the Borici-Creutz action describes two flavors, the number of flavors becomes double i.e.,  $N_f = 2N = 4$  for an action with  $(M^\dagger M)$ . The minimally doubled fermions have physical dispersion relation for  $0 < c_3 < 0.59$  and  $3.41 < c_3 < 4$  and the two chiral phase boundaries are near  $c_3 = 0$  and  $c_3 = 4$  [9]. For the mass spectroscopy, we consider here  $c_3 = 0.1$  i.e., in the region with physical dispersion relation with minimal doubling and reasonably away from the phase boundaries. With pseudofermions the action becomes ,

$$S = \phi^\dagger (M^\dagger M)^{-1} \phi + \beta(\sigma^2 + \pi_\Gamma^2),\tag{7}$$

where,  $\beta = 1/g^2$ . We perform hybrid Monte Carlo(HMC) simulation using this lattice action. The configurations are generated by considering step-size( $\Delta t$ )=0.1 in the leapfrog method and ten steps per trajectory in the molecular dynamics chain. We do not use any preconditioning during the simulation. First 1000 ensembles are rejected for thermalization and analysis is performed over the next 8000 ensembles.

## A. Correlators

For meson mass spectrum calculation, we need to evaluate the correlators

$$C_{ij}(t) = \langle O_i(t) O_j^\dagger(0) \rangle.\tag{8}$$

Since we cannot have orbital angular momentum in 2D, the interpolators ( $O_i$ ) are labelled by parity and charge conjugation only. It is important to choose the appropriate operators which have good overlaps with the low lying states. For the meson spectroscopy, we consider only the odd parity interpolators. The even parity interpolators are not considered as they do not decay exponential and correspond to condensates[15]. Since under parity  $\psi(x, t) \rightarrow \gamma_2 \psi(-x, t)$ , the odd parity interpolators can be constructed with  $\gamma_1$  or  $\gamma_5$ . Along with the local source, one can also construct the interpolators with the fields at different lattice sites shifted along the spatial direction ie., with  $\psi(x \pm n, t)$ . If one considers a relative negative sign in between  $\psi(x + n, t)$  and  $\psi(x - n, t)$  then this corresponds to a derivative source which are found to be important for excited state spectroscopy[15, 22]. Combining the field operators at different lattice sites, many interpolators can be constructed but it was found in our numerical analysis that they mostly couple to the ground state. In[15], a set of nine different interpolators were listed. Here we list some of the parity odd interpolators for the GN model which we expect to couple to ground state as well as excited states:

$$\begin{aligned}
O_1(t) &= \bar{\psi}(x, t) \gamma_5 \psi(x, t) \\
O_2(t) &= \frac{1}{4} \left( (\bar{\psi}(x + m, t) - \bar{\psi}(x - m, t)) \gamma_5 (\psi(x + n, t) - \psi(x - n, t)) \right), (m = 3, n = 3) \\
O_3(t) &= \frac{1}{4} \left( (\bar{\psi}(x + m, t) - \bar{\psi}(x - m, t)) \gamma_5 (\psi(x + n, t) - \psi(x - n, t)) \right), (m = 5, n = 3) \quad (9) \\
O_4(t) &= \frac{1}{4} \left( (\bar{\psi}(x + m, t) - \bar{\psi}(x - m, t)) \gamma_1 (\psi(x + n, t) - \psi(x - n, t)) \right), (m = 4, n = 3) \\
O_5(t) &= \frac{1}{4} \left( (\bar{\psi}(x + m, t) + \bar{\psi}(x - m, t)) \gamma_1 (\psi(x + n, t) - \psi(x - n, t)) \right), (m = 5, n = 3),
\end{aligned}$$

where sum over  $x$  is implied in order to have zero momentum states and  $\gamma_5 = i\gamma_1\gamma_2$ . All the interpolators are odd under  $C$ -parity ( $C = -1$ ). With different values of  $m$  and  $n$ , we can have different interpolators, but the one that are found to couple with ground state as well as the excited states are for the values listed above in Eq.(9), other interpolators do not couple to new states but only reproduce the similar results.

## B. Effective mass calculation

The effective masses are extracted from the correlators at different time slices by the formula

$$M_{eff} = \ln \left( \frac{c(t)}{c(t+1)} \right). \quad (10)$$

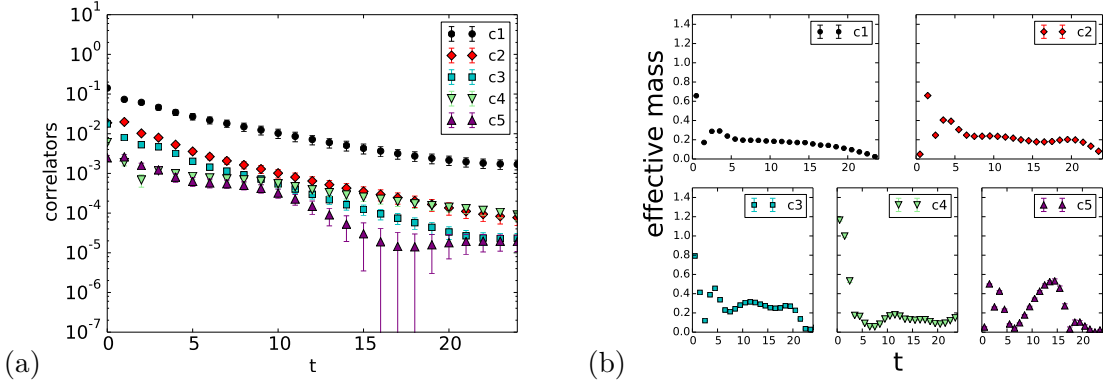


FIG. 1: Diagonal correlators (in the plots  $c1 \equiv C_{11}$ , etc.) and effective mass of meson in GN model for  $16 \times 48$  lattice

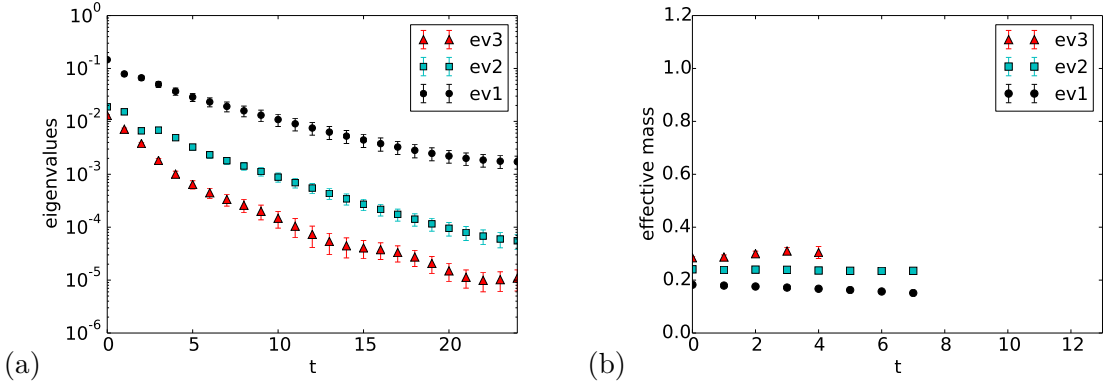


FIG. 2: Eigenvalues and effective mass of the correlators for  $16 \times 48$  lattice

The diagonal correlators  $C_{ii}$  and corresponding effective masses are shown in Fig.1(a) and (b) for  $m_0 = 0.03$  and  $\beta = 0.7$ . The small value of mass is taken to be close to the massless limit. Eq.(10) is an approximate formula and found good for ground state but can also produce approximate values for the excited states. As can be seen from Fig. 1(b), except for the ground state, this procedure cannot extract the excited states well enough, we only see a hint of two other possible states. Analysis of the eigenvalue spectrum of the correlation matrix, on the other hand, provide a better picture for the meson spectroscopy. In the variational method [23, 24], to get the mass spectrum from eigenvalues one solves the generalized eigenvalue problem defined by

$$C(t)\vec{v}^{(n)} = \lambda^{(n)}(t)C(t_0)\vec{v}^{(n)} \quad (11)$$

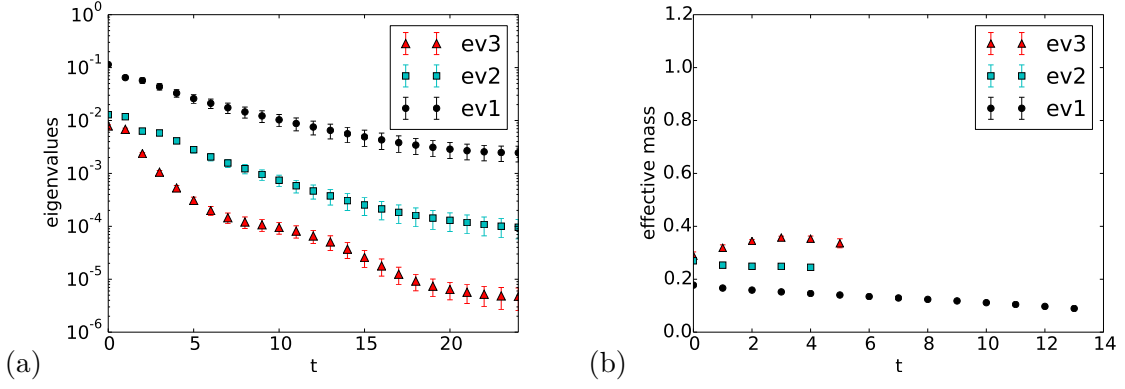


FIG. 3: Eigenvalues and effective mass of the correlators for  $20 \times 48$  lattice

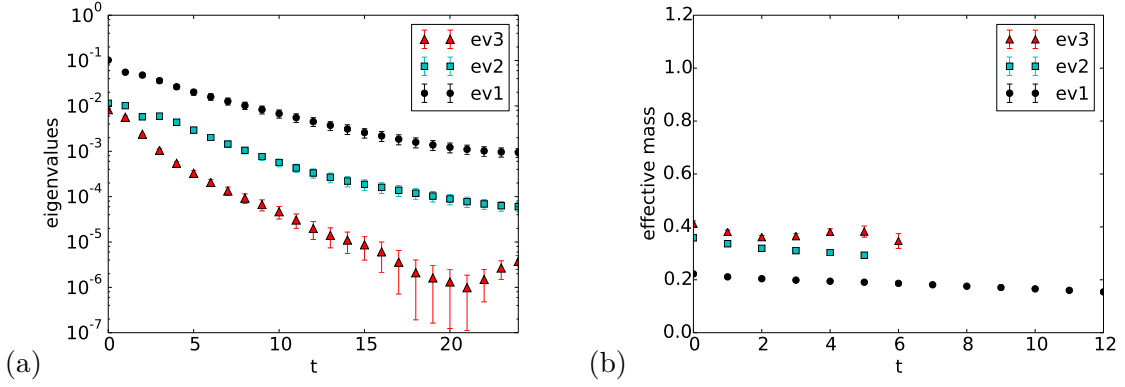


FIG. 4: Eigenvalues and effective mass of the correlators for  $22 \times 48$  lattice

where  $C(t)$  is the  $N \times N$  correlation matrix constructed from  $N$  interpolators  $O_i$ , ( $i = 1, 2, \dots, N$ ). The  $n$ -th eigenvalue behaves as

$$\lambda^{(n)}(t) = e^{-(t-t_0)E_n} \left[ 1 + \mathcal{O}(e^{-(t-t_0)\Delta_n}) \right], \quad (12)$$

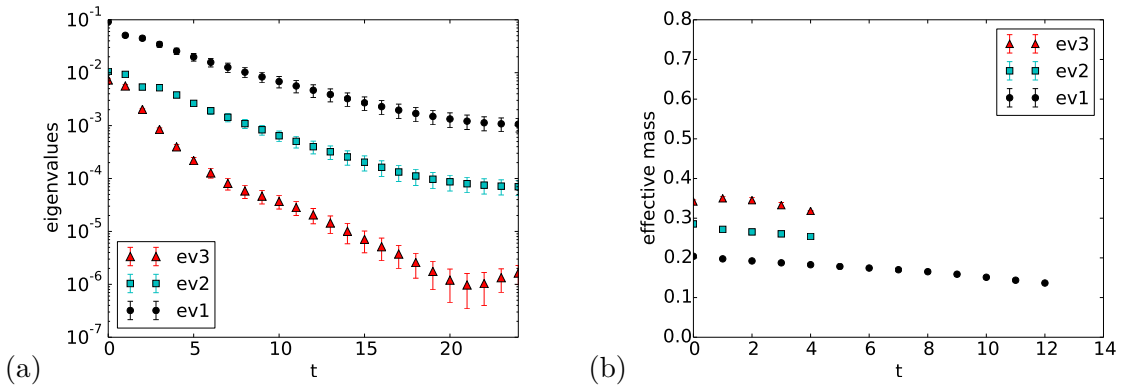


FIG. 5: Eigenvalues and effective mass of the correlators for  $24 \times 48$  lattice

where  $E_n$  is the energy of the  $n$ -th state and  $\Delta_n$  is the energy gap between the neighboring states. In Fig.2 we show the eigenvalue and effective mass plots by solving the generalized eigen value problem for  $16 \times 48$  lattice with  $O_1$ ,  $O_2$  and  $O_3$  interpolators. We are unable to get any extra stable mass values by increasing the matrix dimension of the correlator basis. So, we work with only  $O_1$ ,  $O_2$  and  $O_3$ . The results become noisier if the matrix dimension is increased by including more correlators. We have shown the eigenvalues and effective mass plots for the  $16 \times 48$  lattice in Fig.2. Three mass plateau can be observed in Fig.2(b). In Fig.3 , Fig.4 and Fig.5 we have shown the eigenvalues and effective masses for  $20 \times 48$ ,  $22 \times 48$  and  $24 \times 48$  respectively.. In Fig.6, the volume dependence of the effective

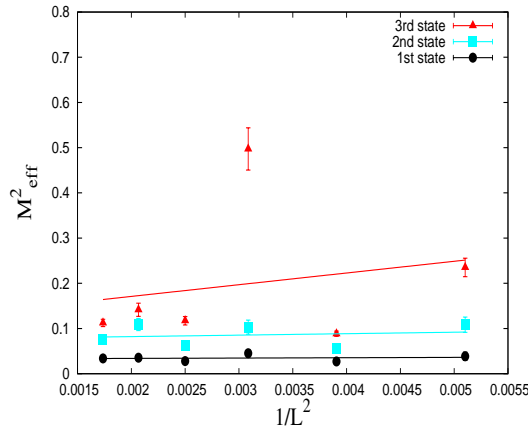


FIG. 6: Volume dependence of the effective mass.

masses are shown. The ground state and the first excited state show no volume dependence and hence can be considered as bound states. The second excited state however shows volume dependence. Specially, for  $18 \times 48$  lattice size, we get an anomalously large mass for the second excited state. The fit for the second excited state shown in Fig.6 includes this anomalous point. In general, scattering states show strong volume dependence and increase linearly with  $1/L^2$ , the volume dependency of the second excited state in our case is not very conclusive. But looking at the fit of the points we expect it to be a scattering state. The results can be contrasted with [15], where except the ground state, all the excited states show strong volume dependence and are scattering states.



### C. Fermion mass and chiral phase transition

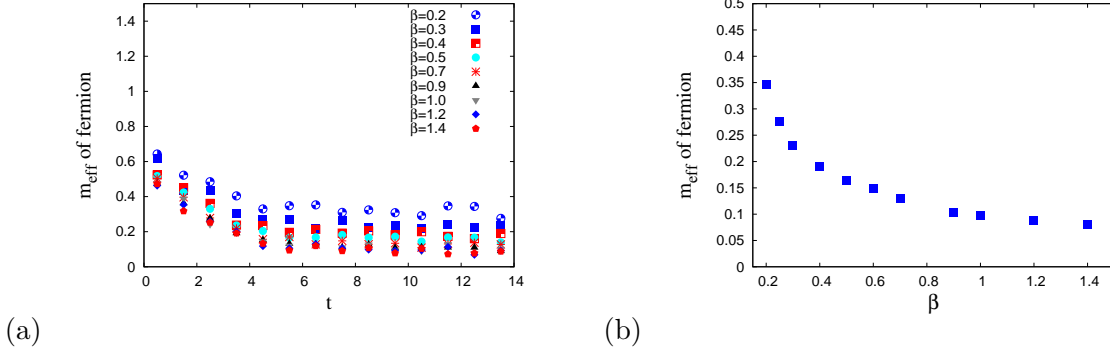


FIG. 7: (a) Effective mass of the fermion in GN model, (b) the variation of effective electron mass with  $\beta$  is consistent with the chiral phase transition demonstrated in Ref.[9]. The fermion bare mass is  $m_0 = 0.03$ .

In GN model, we have also extracted the effective mass of the fermion. For the fermion mass calculation, we consider the correlator with  $O(t) = \psi(x, t)$  and evaluate the effective fermion mass for different values of the coupling constant  $\beta$ . The effective masses for different  $\beta$  are shown in Fig.7(a). We take  $c_3 = 0.0001$  to be close to the phase boundary. Fig. 7(b) shows the variation of effective electron mass with the coupling constant. At small  $\beta$  (i.e., at large coupling), the electron mass rapidly increases indicating a phase transition as can be seen from Fig. 7(b). In [9], it was shown with Borici-Creutz fermion that the GN model with a discrete chiral symmetry shows a second order chiral phase transition at  $\beta \approx 0.4$ . The current result for the  $\beta$ -dependence of the electron mass is consistent with that finding.

### III. MESON IN 2D QED

In this section, we extend the study of spectroscopy with BC fermion formulation to gauge theory. For this purpose, we implement the BC fermion in a 2D  $U(1)$  gauge theory and extract the meson masses. QED in 2D is also a confined theory and serves as a good toy model for QCD. The Gross-Neveu model, having a discrete chiral symmetry undergoes spontaneous breaking, but in the massless Schwinger model, the chiral symmetry is continuous

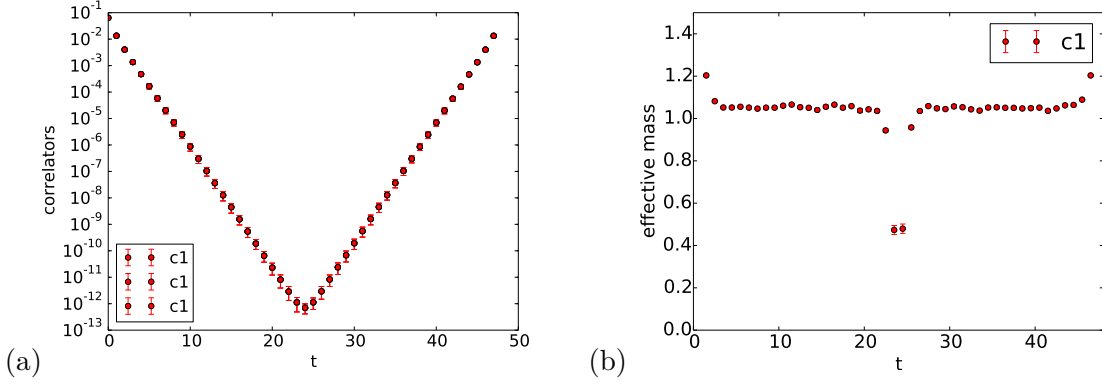


FIG. 8: Effective mass of meson in 2D QED for  $m_0 = 0.05$  and  $\beta = 0.3$ .

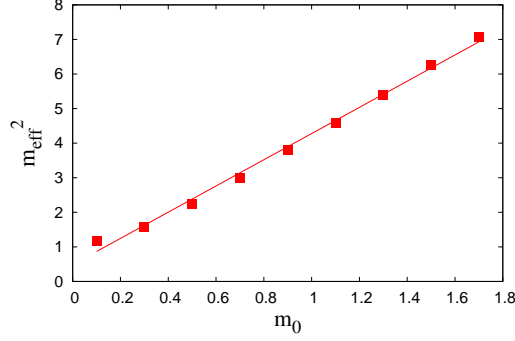


FIG. 9: Fermion mass dependence of  $m_{eff}^2$  in QED<sub>2</sub> for a fixed  $\beta = 0.3$ .

and cannot be spontaneously broken. The lattice action with BC fermion reads,

$$S = \beta \sum_p [1 - \frac{1}{2}(U_p + U_p^\dagger)] + \phi^\dagger (D^\dagger D)^{-1} \phi. \quad (13)$$

where  $U_p$  is the Wilson Plaquette action with

$$U_p = U_{i,\mu} U_{i+\mu,\nu} U_{i+\nu,\nu}^\dagger U_{i,\nu}^\dagger. \quad (14)$$

where,  $i$  is the site index and  $\mu, \nu$  are the directions and  $D$  is the BC Dirac operator defined in eqn.(6). After including gauge fields we get,

$$D_{mn} = \frac{1}{2}(\gamma_\mu + i(\Gamma - \gamma_\mu))U_\mu(n - \mu)\delta_{n,m+\mu} - \frac{1}{2}(\gamma_\mu - i(\Gamma - \gamma_\mu))U_\mu^\dagger(n)\delta_{n,m-\mu} - ((2 - c_3)i\Gamma - m_0)\delta_{m,n}. \quad (15)$$

Here we concentrate only for the lowest lying meson mass. The correlator with operator  $O_1(t)$  couples to the ground state and provides the mass for the lowest state. In Fig.8(a)

we have shown the correlator at different time slices and the effective meson mass in 2D QED. The results are presented for the fermion (electron) mass  $m_0 = 0.05$  and  $\beta = 0.7$ . The ground state mass  $m_{eff} \approx 1.0$  and is much larger than  $(2m_0)$ . The square of meson mass ( $m_{eff}^2$ ) shows a linear dependence on the fermion mass as illustrated in Fig.9. For light fermions, the meson mass is much larger than twice the bare fermion mass  $2m_0$ . As the mass increases, the available phase space decreases and the contribution to the meson mass from interaction diminishes so the difference  $(m_{eff} - 2m_0)$  becomes smaller. As can be seen from Fig.9, for heavy fermions, the meson mass becomes less than  $2m_0$ . This can be explained from the fact that for heavy fermions, the quantum corrections to the effective mass become small as explained above, but the binding energy due to strong coupling is still large, so in combination of these two, the effective mass become less than the sum of individual particles as one observes for atomic or nuclear mass where the mass of the atom/nucleus is less than the sum of the individual constituent masses.

#### IV. SUMMARY

Minimally doubled fermions may provide an efficient lattice formalism to study chiral fermion which is expected to be computationally cheaper than the other existing lattice formalisms. Since, both the minimally doubled fermion formulations (KW and BC) break hypercubic symmetries on the lattice, they require non-covariant counter terms. Only detailed numerical studies can confirm how bad or manageable its effects are on the lattice, and whether any meaningful computation is possible with minimally doubled fermion or not. In this work, we have studied the BC fermion in some simple models. We have extracted the excited state mass spectrum in Gross-Neveu model using BC fermion. Interestingly, the absence of volume dependence of the states suggests that they are all mesonic bound states and no scattering states are found. The effective fermion mass has also been evaluated as a function of the coupling constant  $\beta$ . It shows a phase transition consistent with the result obtained in [9]. We have also evaluated the lowest lying meson mass in QED<sub>2</sub>. For light fermion, the meson is much heavier than  $2m_0$  and for heavy fermion the meson mass becomes less than  $2m_0$  as the renormalized fermion mass becomes much smaller than the bare mass at strong coupling. Our investigations suggest that BC fermion formalism might be a promising alternative to study the chiral fermions on a lattice. One obviously needs

more detailed numerical study in 4D gauge theory with dynamical BC fermion to confirm that claim.

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- [1] L.H. Karsten, Phys. Lett. B **104**,315 (1981).
  - [2] F. Wilczek, Phys. Rev. Lett. **59**,2397 (1987).
  - [3] M. Creutz, JHEP **0804**, 017(2008); Pos LAT2008, 080 (2008).
  - [4] A. Borici, Phys. rev. D. **78**, 074504 (2008).
  - [5] P. F. Bedaque, M. I. Buchoff, B.C. Tiburzi, A. Walker-Loud, Phys. Lett. B. **662**,449 (2008).
  - [6] S. Capitani, J. Weber, H. Wittig, Phys. Lett. B681, 105 (2009).
  - [7] S. Capitani, M. Creutz, J. Weber, H. Wittig, JHEP **1009**, 027 (2010).
  - [8] D. Chakrabarti, S. J. Hands, A. Rago, JHEP **0906**, 060 (2009).
  - [9] J. Goswami, D. Chakrabarti and S. Basak, Phys. Rev. D **91**, no. 1, 014507 (2015).
  - [10] M. Creutz, T. Kimura, T. Misumi, Phys. Rev. D.**83**, 094506 (2011).
  - [11] T. Misumi, JHEP **1208**, 068 (2012).
  - [12] T. Korzec, F. Knechtli, U. Wolff, B. Leder, PoS LAT2005, 267 (2006).
  - [13] S. J. Hands, C. Strouthos, Phys. Rev. B**78**, 165423 (2008), W. Armour, S. Hands, C. Strouthos, Phys. Rev. B**81**, 125105(2010).
  - [14] Y. Araki, PoS LAT2011, 054 (2011); Phys. Rev B **85**, 125436 (2012).
  - [15] J. Danzer and C. Gattringer, PoS LAT **2007**, 092 (2007).
  - [16] C. Gutsfeld, H. A. Kastrup and K. Stergios, Nucl. Phys. B **560**, 431 (1999).
  - [17] C. Gattringer, I. Hip, C. B. Lang, Phys. Lett. B **466**,287 (1999).
  - [18] K. Cichy, A. Kujawa-Cichy and M. Szyniszewski, Comput. Phys. Commun. **184**, 1666 (2013).
  - [19] L. Giusti, C. Hoelbling and C. Rebbi, Phys. Rev. D **64**, 054501 (2001).
  - [20] W. Bietenholz, I. Hip, S. Shcheredin and J. Volkholz, Eur. Phys. J. C **72**, 1938 (2012).
  - [21] S. J. Hands, A. Kocić, J. B. Kogut, Nucl. Phys. B**390**, 355 (1993), Ann. Phys. **224**, 29 (1993).
  - [22] C. Gattringer, L. Y. Glozman, C. B. Lang, D. Mohler and S. Prelovsek, Phys. Rev. D **78**, 034501 (2008).
  - [23] C. Michael, Nucl. Phys. B **259**, 58 (1985).
  - [24] M. Luscher and U. Wolff, Nucl. Phys. B **339**, 222 (1990).